Amplitude in MICE Step IV

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Categories of phase-space presented

- **Toy Monte Carlo**
  - Takes any input distribution (x, y, p_x, p_y, p_z) (see **A**)
  - Deterministic Bethe-Bloch energy loss for given Toy absorber
  - Gaussian scattering $\mathcal{N}(0, \theta_0)$, $\theta_0 = \frac{13.6}{pc\beta}\sqrt{x/X_0}(1 + 0.038\ln(x/X_0))$
  - Measures the beam directly "downstream" of the absorber
- **MAUS Monte Carlo Simulation**
  - Generates beam of input normalised emittance ϵ_i and momentum p_i inside the upstream solenoid, given the field
  - Passes through the simulated TKU and TKD stations, the MICE absorber (LiH currently) and a virtual plane every 5 cm.
  - Standard physics processes
- **MICE Data**
  - Beam sampled in the hall at every TKU and TKD stations
  - Particle species selection currently using TOF01
  - No transmission selection in the analyses, but 140 ± 5 MeV/c input
  - **2016/04 setting 1.2** (3–10 mm, 140 MeV/c, 880 mm $\beta_\perp$)

Transverse single-particle amplitude

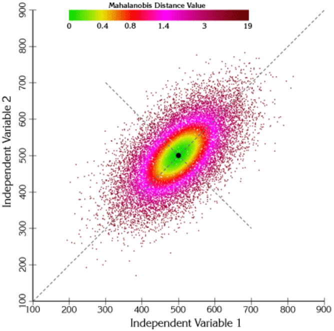
The **transverse amplitude** is defined as:

$$A_\perp \equiv \epsilon_n u_i^T \Sigma^{-1} u_i \tag{1}$$

with Σ the covariance matrix and u_i the phase-space column vector of the i^{th} particle. The u_i are centred so that $u_{i,\alpha} = \alpha_i - \langle\alpha\rangle$. For a gaussian beam, the amplitudes are distributed as a χ^2 distribution with 4 degrees of freedom. Its mean is $\langle A_\perp\rangle = \epsilon_n\langle\chi_4^2\rangle = 4\epsilon_n$.

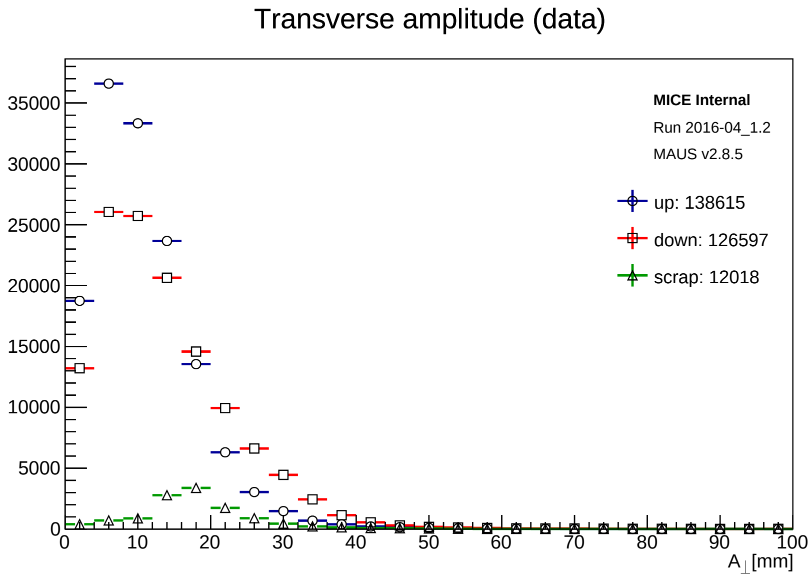
The amplitude gives a definition of a weight to select **any given fraction α of the beam**, rejecting the tails if need be. It is analogous to the squared **Mahalanobis distance**, d^2, scaled by the emittance, ϵ_n.

This basically represents the **Euclidean distance** of the particle to the centre of the distribution, in a **metric defined by $\boldsymbol{\Sigma}$**.



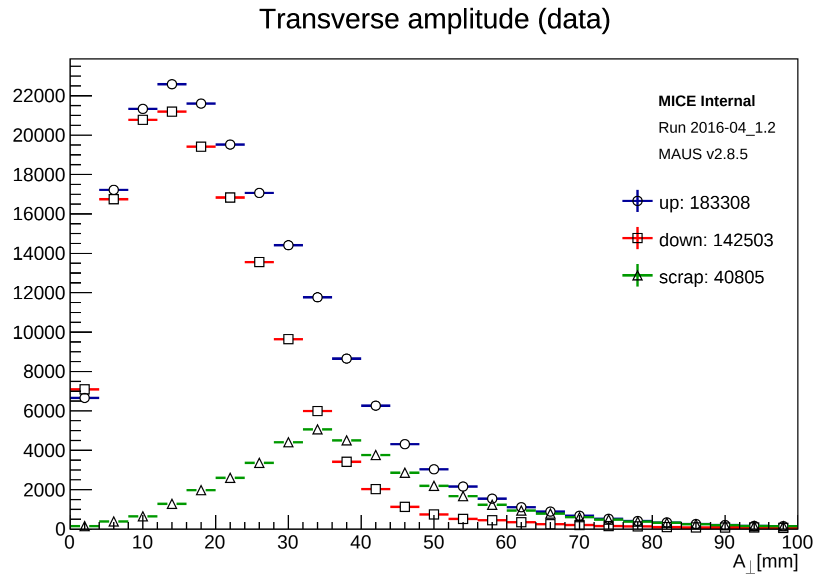
Amplitude evolution in the 3 mm beam

Data



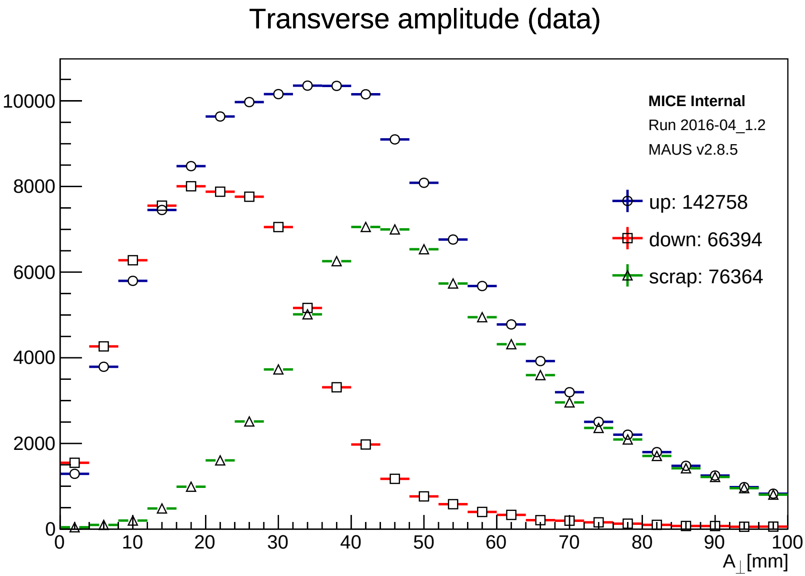
Amplitude evolution in the 6 mm beam

Data



Amplitude evolution in the 10 mm beam

Data



Corrected amplitude

To remove the tail effects from the amplitude distribution, one can only consider particles with amplitudes smaller than the one being computed:

1. Compute Σ for the whole sample
2. Compute the amplitudes and use it to order the particles
3. Add the largest amplitude to the distribution
4. Decrement $\Sigma, \vec{\mu}$ by removing the particle with largest amplitude†
5. Repeat at (2) until all the particles amplitudes have been produced

$$\dagger\, \Sigma_{\alpha\beta} = \frac{n-1}{n-2}\Sigma_{\alpha\beta} - \frac{n}{(n-1)(n-2)}(\alpha_n - \langle\alpha\rangle)(\beta_n - \langle\beta\rangle) \tag{2}$$

$$\dagger\, \mu_\alpha = \frac{1}{n-1}\left(n\mu_\alpha - \alpha_n\right) \tag{3}$$

Assuming that our beam has preserved a somewhat Gaussian core, this method allows to reproduce reliably the core amplitudes.

Effect of the correction on amplitude

Taking only the particles with smaller amplitudes have no effect on the amplitude distribution for a Gaussian beam and correct the distribution for other distributions.

Removing high amplitudes effectively truncate the distribution. This scales the covariance matrix but preserves its shape, i.e.

$$\Sigma' \to (\lambda I)\Sigma \tag{4}$$

As a result, the amplitude reads

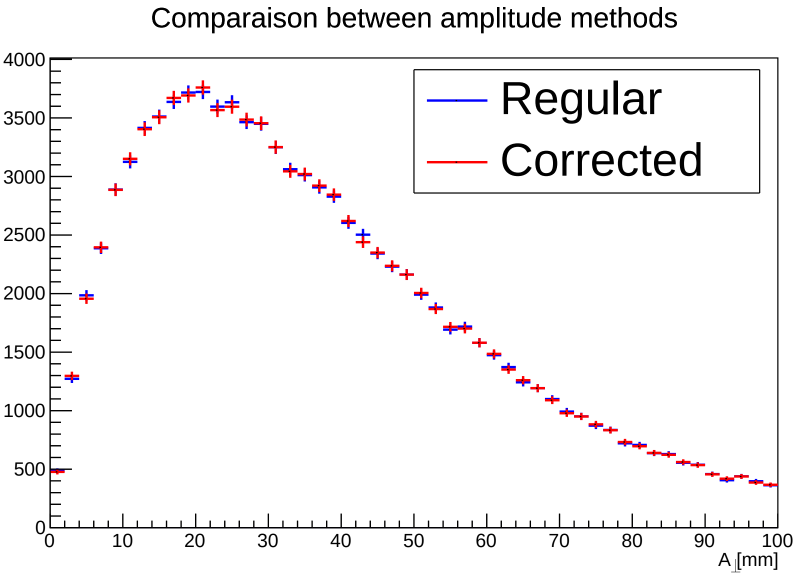
$$\Sigma^{-1} \to \frac{1}{\lambda}\Sigma^{-1}, \qquad |\Sigma| \to \lambda^d|\Sigma| \tag{5}$$

$$A'_\perp = \frac{1}{m}\sqrt[d]{\lambda^d|\Sigma|}\frac{1}{\lambda}\boldsymbol{x}^T\Sigma^{-1}\boldsymbol{x} = \frac{1}{m}\sqrt[d]{|\Sigma|}\boldsymbol{x}^T\Sigma^{-1}\boldsymbol{x} = A_{perp} \tag{6}$$

$\rightarrow$ All we want to achieve is the **right shape** of the RMS ellipse, the amplitudes themselves are **scale-invariant** because of the ϵ_n term.

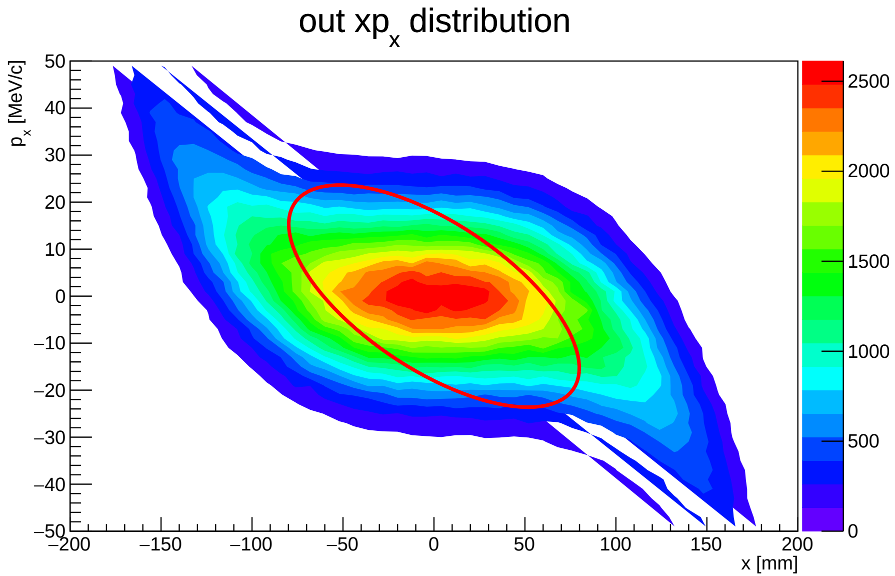
Corrected amplitude in a 4D Gaussian beam

Toy



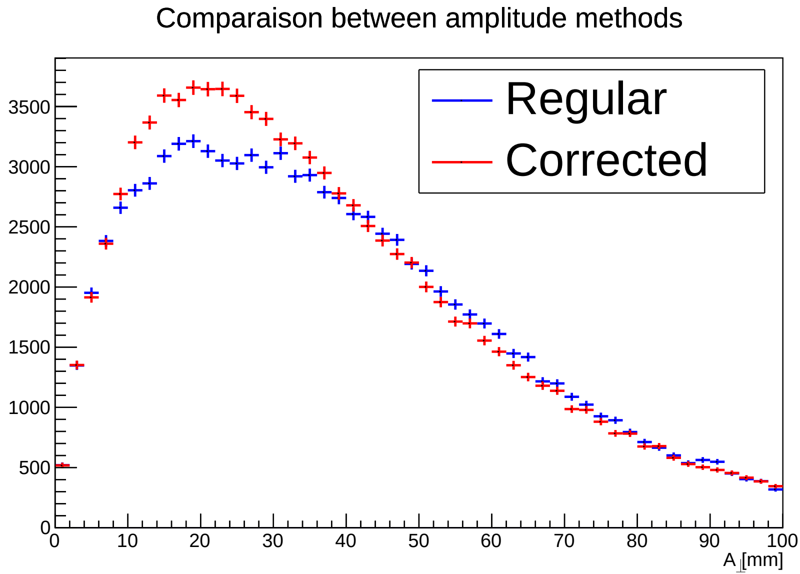
Non-linear distribution profile in 2D

Toy

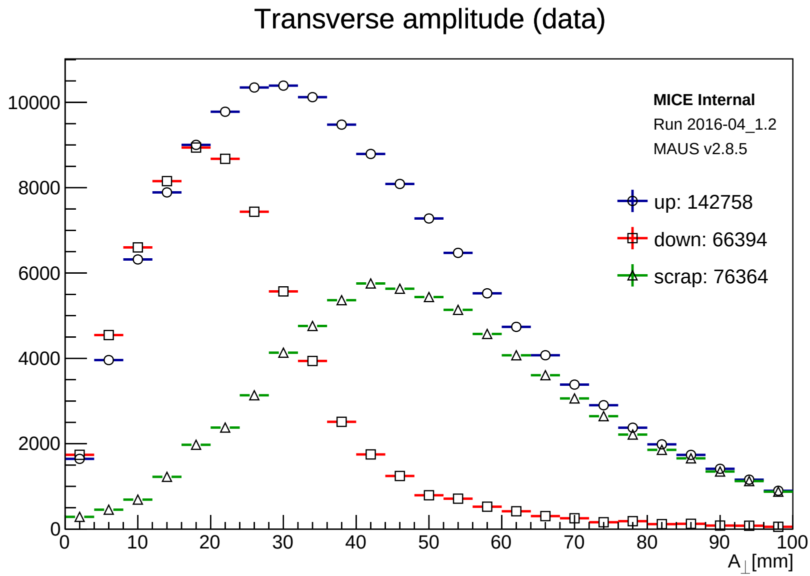


Corrected amplitude in a 4D non-linear beam

Toy



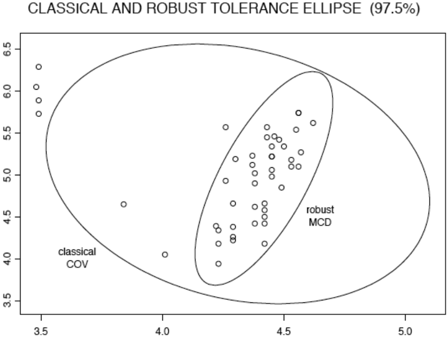
Corrected amplitude evolution in the 10 mm beam **Data**



Robust Mahalanobis distance

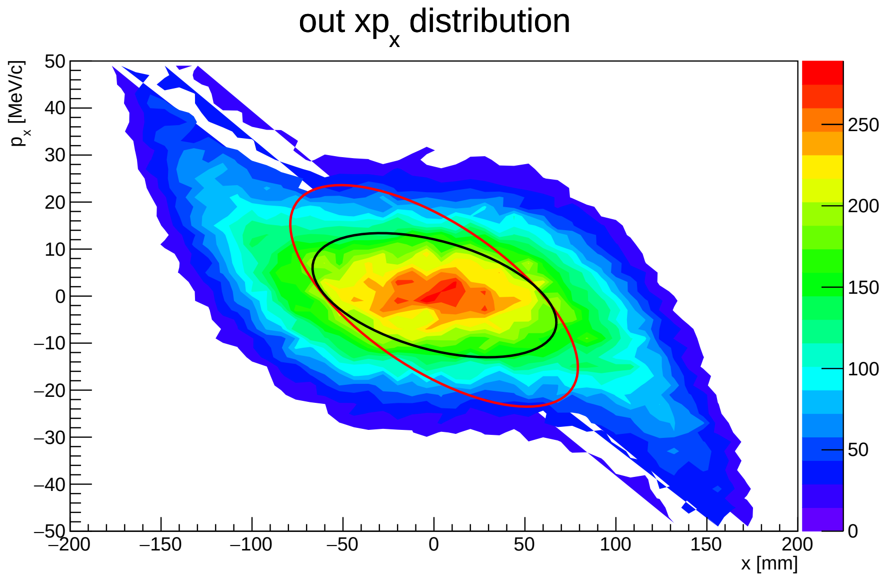
The **Mahalanobis distance**, on which the amplitude computation is based, relies heavily on the quality of the covariance matrix estimation, which is poor in the presence of a lot of outliers.

The **Minimum Covariance Determinant** (MCD) estimator is the most common way to deal with it. It finds the the subset of points (fraction α) that has the lowest covariance determinant, effectively getting rid of outliers.



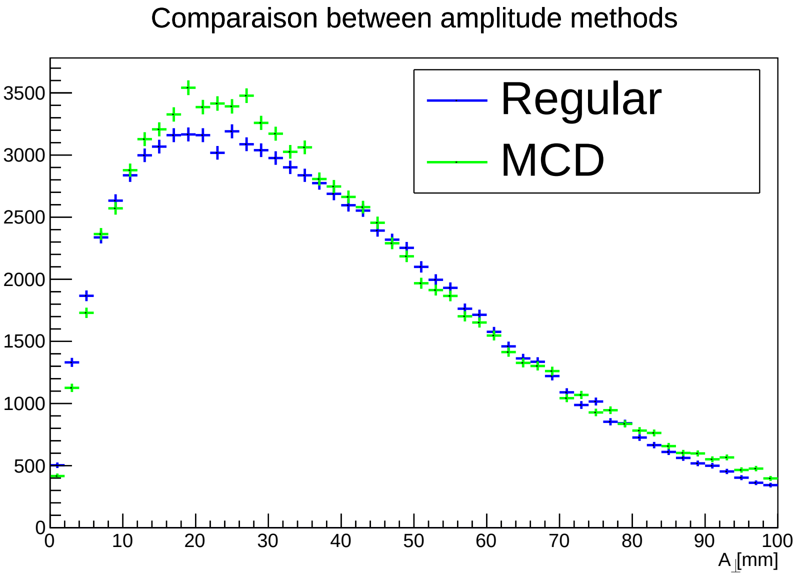
Adjustment of the RMS ellipse with MCD

Toy



Corrected amplitude in a 4D non-linear beam

Toy



Excess level and significance

The level of the excess can be caracterised bin by bin or for a certain range of bins in “blind” analysis (choose range from the data) by

$$\mathcal{E} = \frac{N_D - N_U}{N_U}. \tag{7}$$

The significance of a measured excess can be quantified by the p-value of the null hypothesis[1] (phase-space volume conserved). For a simple counting of amplitudes in MICE, with Poisson statistics, one gets

$$p = \sum_{n=N_D}^{+\infty} \frac{e^{-N_U} N_U^n}{n!}. \tag{8}$$

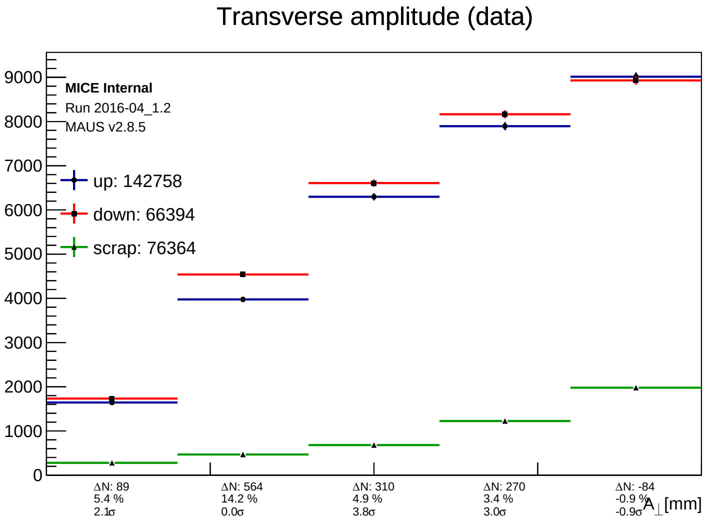
Finally, given the p-value, the significance is given in amounts of σ as

$$\mathcal{S} = \sqrt{2}\,\mathrm{erf}^{-1}(1-2p). \tag{9}$$

[1] Signal Significance in Particle Physics, P.K. Sinervo

Significance of the excess (10 mm, no syst.)

Data



→ For the range $A_\perp \in [0, 20]$, $\Delta N/N =$ **4 %**, i.e. a **6.68$\boldsymbol{\sigma}$** excess

Using amplitudes to produce a subsample

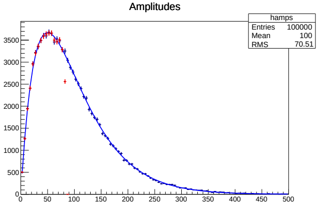
The method to produce an α-subsample is as follows:

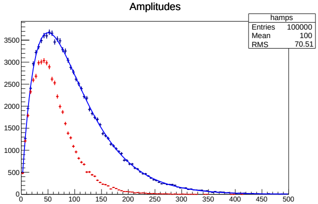
1. calculate the amplitudes, $A_\perp^i$, $i = 1, \ldots, N$, of every particle;
2. find a limit $A_\perp^\alpha$ so that the sample of all particles that verify $A_\perp < A_\perp^\alpha$ represents a fraction α of the entire population;
3. re-evaluate the covariance matrix Σ on the reduced sample;
4. repeat 1., 2. and 3. until we get convergence on the sample.

$\rightarrow$ The RMS emittance of the subsample is the **subsample emittance**

$\rightarrow$ The volume occupied by the subsample **fractional emittance**

$\rightarrow$ Must select the **same amount of particles** up and downstream





Amplitude based fractional quantities

Maximum $A_\perp$

Corresponds to the maximum amplitude in the selected sample, i.e. that of an **ellipsoid encompassing the sample** going through the particle of largest amplitude.
We have that

$$\frac{\Delta A_\perp^M}{A_\perp^M} = \frac{\Delta \epsilon_n}{\epsilon_n} \qquad (10)$$

for a Gaussian beam (similar ellipsoids).

Subsample ϵ

Corresponds to the **emittance of the selected sample**.
We have that

$$\frac{\Delta e_\alpha}{e_\alpha} = \frac{\Delta \epsilon_n}{\epsilon_n} \qquad (11)$$

for a Gaussian beam (similar ellipsoids). More reliable than $A_\perp^M$ as less variable.

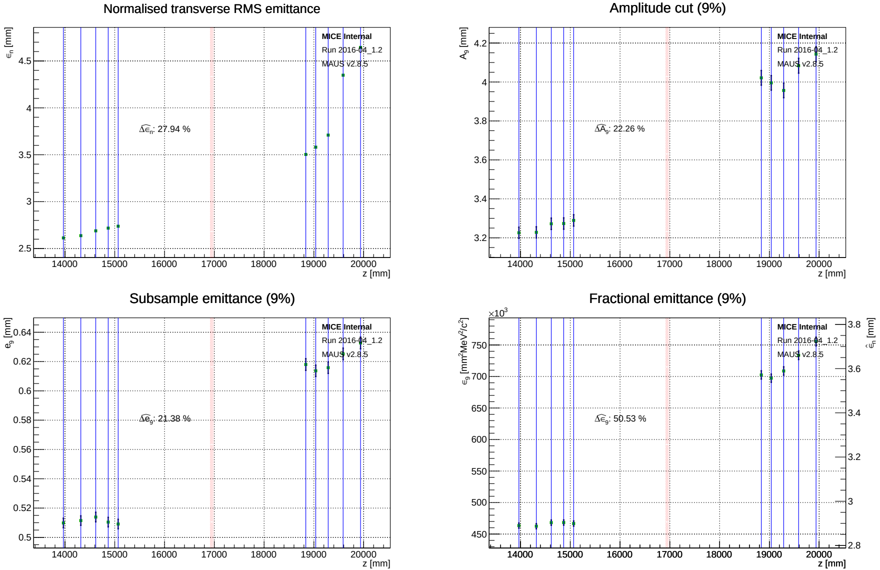
Fractional ϵ

Corresponds to the **phase-space volume occupied by the selected sample**. As the set is always convex, computing its hull is the most efficient way.
We have that

$$\epsilon_n = \frac{\sqrt{2\epsilon_\alpha}}{m\pi} \qquad \frac{\Delta e_\alpha}{e_\alpha} \sim 2\frac{\Delta \epsilon_n}{\epsilon_n} \qquad (12)$$

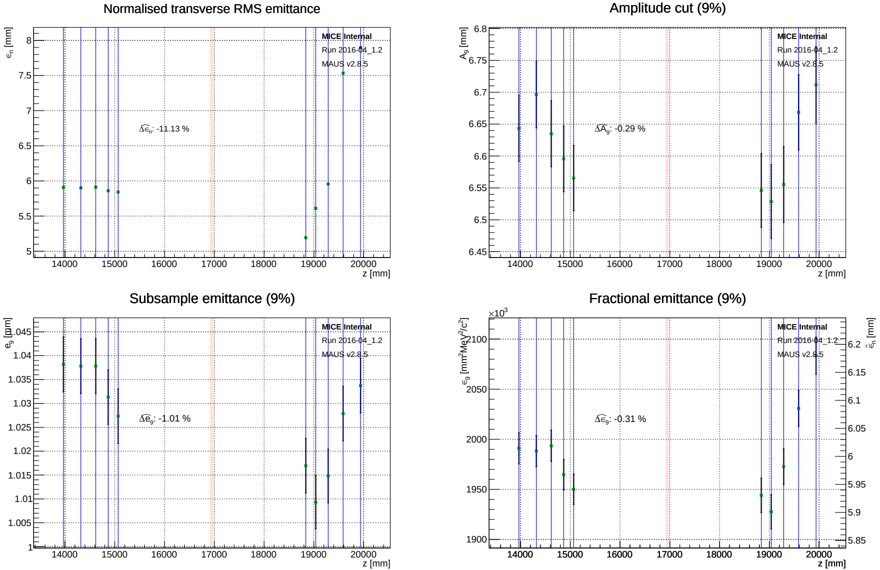
Fractional quantities in the 3 mm beam

Data



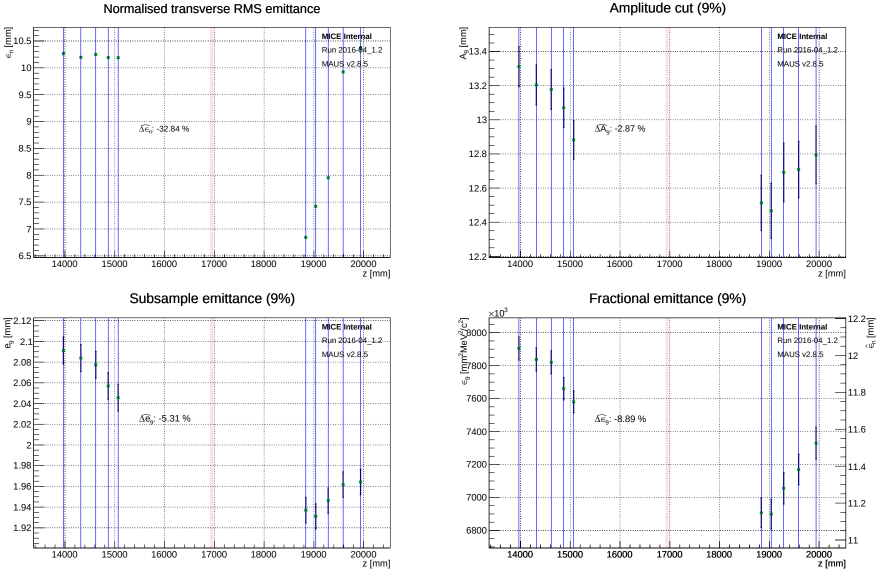
Fractional quantities in the 6 mm beam

Data



Fractional quantities in the 10 mm beam

Data



k-Nearest Neighbours

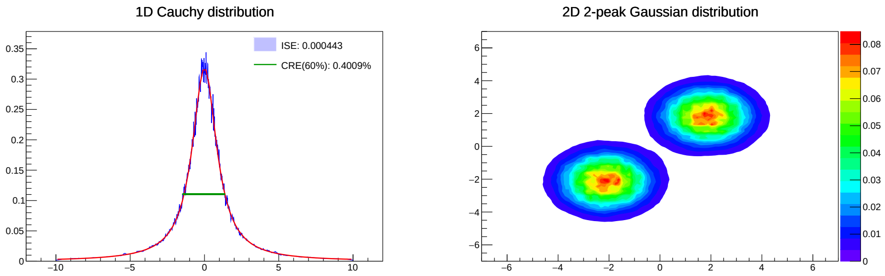
For a given point $\boldsymbol{x}$, find the k closest points in the input cloud, find the distance R_k to the k^{th} point and compute the local density as

$$\rho(\boldsymbol{x}) = \frac{k}{\mathcal{V}_n} = \frac{k\Gamma\left(\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}} R_k^n}, \tag{13}$$

with $\mathcal{V}_n$ the volume of the n-sphere centred in $\vec{x}$ of radius R_k.

Several ways to optimize k

- Rule of thumb, simply fix $k = \sqrt{N}$
- Minimize the information criterion (AIC, BIC)



Connection between amplitude and density

The amplitude basically corresponds to a Mahalanobis distance. The factor $R^2 = \vec{x}^T \Sigma^{-1} \vec{x}$ defines an iso-hypersurface on which $\rho(\vec{x}) = \rho(R)$:

$$\rho(R) = \underbrace{\frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}}}_{\equiv \rho(0)} \exp(-R^2/2) \quad \rightarrow \boxed{R^2 = \ln\left[\frac{\rho^2(0)}{\rho^2(R)}\right]} \tag{14}$$

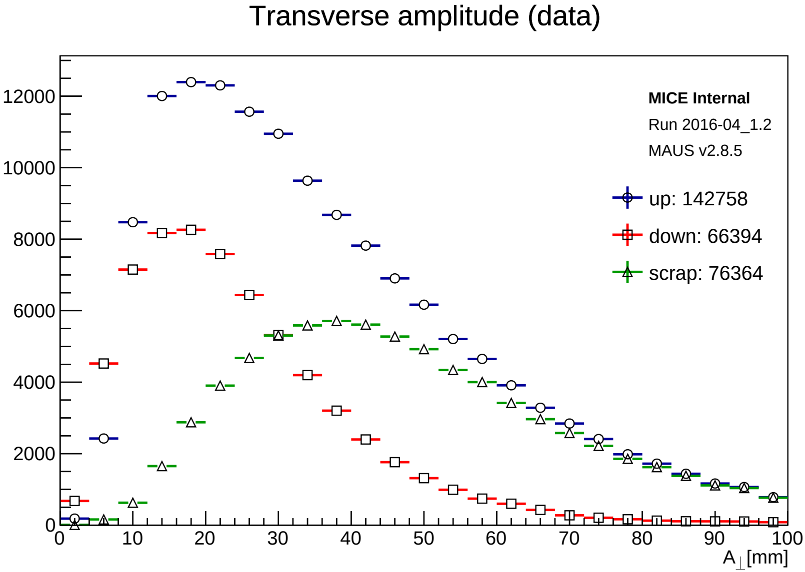
The second element is the prefactor ϵ_n, proportional to the volume, $V_{\mathcal{E}}$, of the RMS ellipse (fraction $\gamma(n/2, 1/2)$ of the sample):

$$V_{\mathcal{E}} = \frac{\pi^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}, \epsilon_n = \frac{1}{m}|\Sigma|^{\frac{1}{d}} \quad \rightarrow \epsilon_n = \frac{1}{m\pi}\left[\Gamma\left(\frac{n}{2}+1\right) V_{\mathcal{E}}\right]^{\frac{2}{n}} \tag{15}$$

The RMS volume $V_{\mathcal{E}}$ can be computed for any distributions. The generalised amplitude thus reads

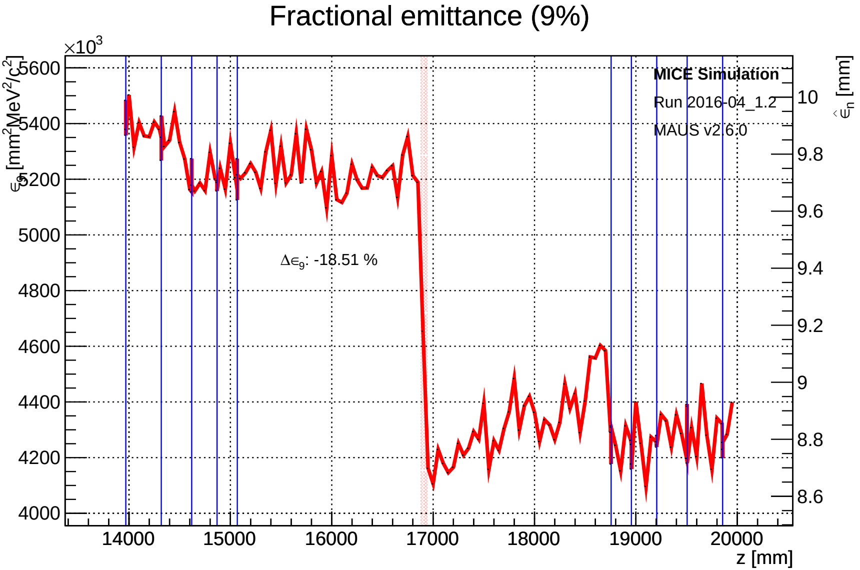
$$\boxed{A_{\perp} = \frac{1}{m\pi}\left[\Gamma\left(\frac{n}{2}+1\right) V_{\mathcal{E}}\right]^{\frac{2}{n}} \ln\left[\frac{\rho^2(0)}{\rho^2(R)}\right]} \tag{16}$$

Generalised amplitude evolution in the 10 mm beam **Data**



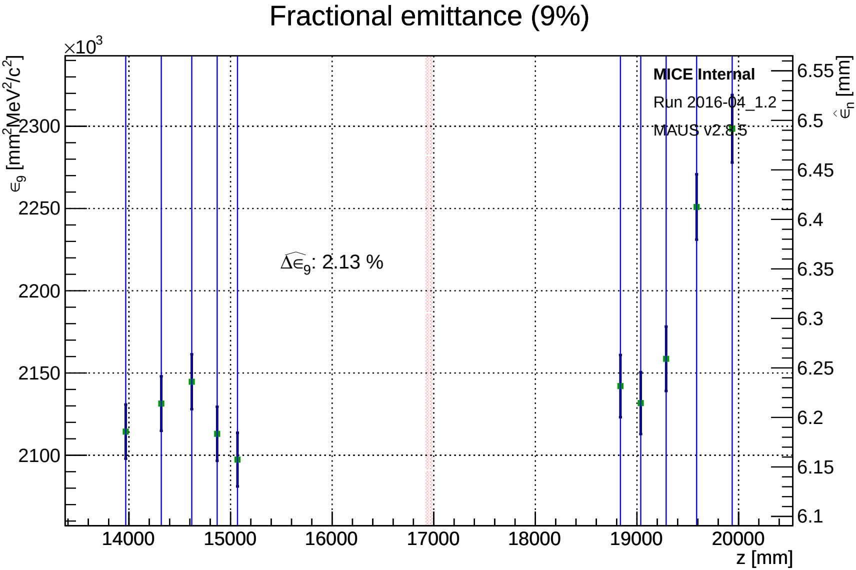
Phase-space volume in a 10 mm beam

Simulation



Phase-space volume in the 6 mm beam

Data



Phase-space volume in the 10 mm beam

Data