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Estimating the muon momentum at the absorber from TOF1 and TOF2

Let s_1 be the path length of a muon from TOF1 to the centre of the absorber.

Let s_2 be path length of a muon from the centre of the absorber to TOF2.

(s_1 and s_2 are the true path lengths, corrected for the angle of the muon wrt the axis.)

Let p be the momentum of a muon at the centre of the absorber – which is the quantity that we want.

Let 2Δ be the total momentum loss of the muon in the absorber.

Assume the absorber is thin (v. good approximation for LiH).

Let t be time of flight between TOF1 and TOF2.

With no absorber the TOF2 - TOF1 time difference (i.e. the total time of flight) is (with $c = 1$)

$$\begin{aligned} t_0 &= s_1 \frac{E}{p} + s_2 \frac{E}{p} \\ &= t_1 + t_2 \,. \end{aligned} \tag{1}$$

When the absorber is present and the total momentum loss of the muon is 2Δ, the total time of flight is exactly

$$t_A = s_1 \frac{\sqrt{(p+\Delta)^2 + m^2}}{(p+\Delta)} + s_2 \frac{\sqrt{(p-\Delta)^2 + m^2}}{(p-\Delta)} \,. \tag{2}$$

This could be solved iteratively for p, given an initial estimate of Δ. For small Δ we can write

$$\begin{aligned} t_A &\approx t_1 + \frac{dt_1}{dp}\Delta + \frac{1}{2}\frac{d^2 t_1}{dt^2}\Delta^2 + t_2 - \frac{dt_2}{dp}\Delta + \frac{1}{2}\frac{d^2 t_2}{dt^2}\Delta^2 \\ &= t_1 + t_2 + (s_1 - s_2)\frac{d(E/p)}{dp}\Delta + \frac{1}{2}(s_1 + s_2)\frac{d^2(E/p)}{dt^2}\Delta^2 \\ &= t_0 - (s_1 - s_2)\frac{m^2}{p^2 E}\Delta + \frac{1}{2}(s_1 + s_2)\frac{m^2}{p^2 E}\left(\frac{p}{E} + 2\frac{E}{p}\right)\frac{\Delta^2}{E} \\ &= t_0 + \delta t \end{aligned}$$

where

$$\delta t = -(s_1 - s_2)\frac{m^2}{p^2 E}\Delta + \frac{1}{2}(s_1 + s_2)\frac{m^2}{p^2 E}\left(\frac{p}{E} + 2\frac{E}{p}\right)\frac{\Delta^2}{E} \,. \tag{3}$$

Surprisingly (or perhaps not) this says that, to first order, if the absorber is exactly at the mid-point of the flight path between TOF1 and TOF2 ($s_1 = s_2$), the time of flight is exactly that of a muon with momentum p, the true momentum at the centre of the absorber. In practice $s_1 - s_2$ is not zero, but will be small compared with $s_1 + s_2$.

A small correction can be applied to an initial estimate of momentum assuming no energy loss, p_0 to obtain a better estimate of the true momentum.

The initial estimate of momentum, assuming no energy loss, can be made from the measured time of flight, t_m, giving

$$p_0 = \frac{m(s_1 + s_2)}{\sqrt{t_m^2 - (s_1 + s_2)^2}}\,.$$

p_0 can then be used to obtain (from the Bethe-Bloch expression or a look-up table) the energy loss at the absorber, $\delta E(p_0)$ and $\Delta(p_0) = E\delta E(p_0)/(2p_0)$ (since I have defined the total momentum loss in the absorber as 2Δ). The corrected momentum is then

$$\begin{aligned}
p_c &= p_0 + \frac{dp}{dt}\delta t \\
p_c &= p_0 + \left(\frac{dt}{dp}\right)^{-1}\delta t \\
&= p_0 - \frac{1}{s_1 + s_2}\frac{Ep^2}{m^2}\delta t \\
&= p_0 + \frac{s_1 - s_2}{s_1 + s_2}\Delta - \frac{1}{2}\left(\frac{p_0}{E_0} + 2\frac{E_0}{p_0}\right)\frac{\Delta^2}{E_0} \\
&= p_0 + \frac{s_1 - s_2}{s_1 + s_2}\Delta - \frac{1}{2}\left(\frac{s_1 + s_2}{t_m} + 2\frac{t_m}{s_1 + s_2}\right)\frac{\Delta^2}{E_0} \\
&= p_0 + \frac{s_1 - s_2}{s_1 + s_2}\Delta - \left(\frac{s_1 + s_2}{t_m} + 2\frac{t_m}{s_1 + s_2}\right)\frac{s_1 + s_2}{2p_0 t_m}\Delta^2\,.
\end{aligned}$$

This could be iterated, but that's probably unnecessary since the correction will be rather small given that both $(s_1 - s_2)/(s_1 + s_2)$ and Δ are small.

Does it work?

I have written a short program to try this. I used an eye-ball fit to the dE/dX versus momentum curve for LiH. Stochastic processes (i.e. Landau fluctuations) were not included. The results for various combinations of s_1 and s_2, and artificially inflated ($\times 2$) dE/dX are shown in Figure 1. The procedure seems to work well enough without iteration even for the inflated energy loss cases and unrealistically asymmetric values of s_1 and s_2: the 'reconstructed' momentum is within 1 MeV/c of the true momentum for $p_\mu >\approx 130$ MeV/c (where the muon is losing a substantial fraction of its initial momentum in the absorber). In the most realistic case (bottom right plot) the agreement is very good. I do not understand the very good agreement shown in the bottom lefthand plot; presumably some cancellations are at work.

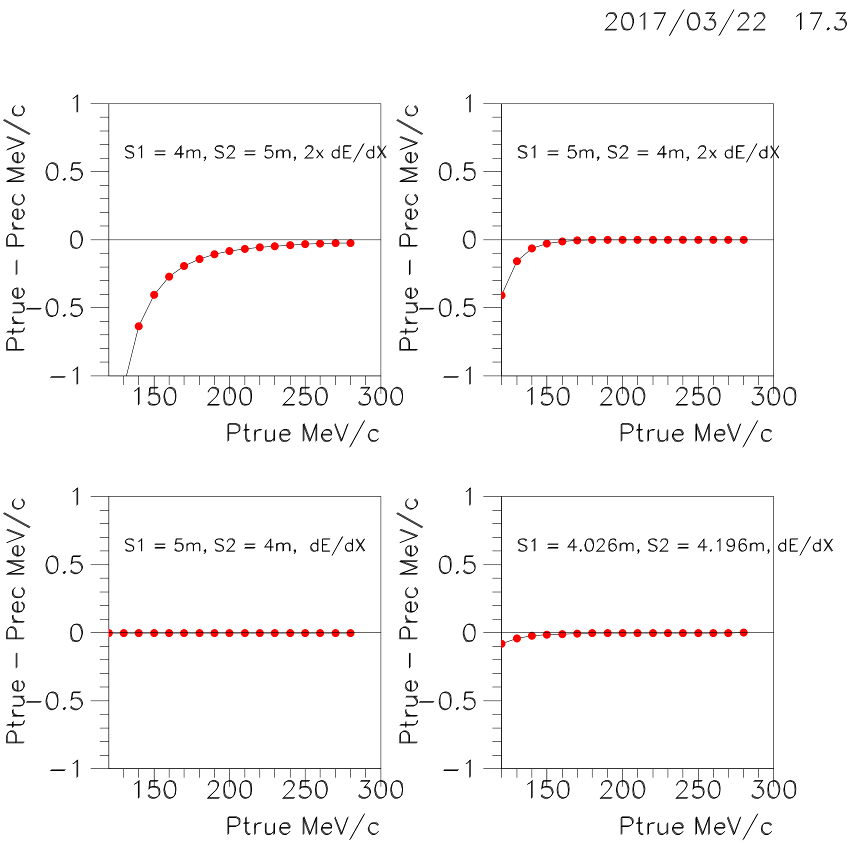

Figure 1: Difference between reconstructed and true muon momentum at the absorber centre for a few case. The bottom right hand plot is the most realistic.

More realistic approach

Chris Rogers pointed out that this model is too simplistic because there is significantly more material between TOF1 and TOF2 than just the absorber: the material in TOF1; the upstream tracker planes; the He window; any safety and absorber windows, and a similar amount of material downstream of the absorber.

It might be possible, although it looks rather ugly, to repeat the algebra including each energy loss component. It is probably simpler to use an iterative procedure *something* like the following:

1. Obtain an initial estimate of momentum, p_0, from the TOF1-TOF2 time of flight, assuming no energy loss;

2. Assuming an initial momentum of p_0 at the centre of the absorber, compute the total time of flight allowing for energy loss at each piece of material;

3. Correct p_0 according to the difference between the computed and measured times of flight, δt using

$$p_0 \to p_0 + \delta p = p_0 + \frac{1}{s_1 + s_2} \frac{Ep^2}{m^2} \delta t \ ,$$

and

4. Repeat from step 2 until there is close enough agreement between measured and computed times of flight.

I haven't tried this (it would require a few rainy days) but I think such a procedure should work. The momentum resolution would ultimately be determined by energy loss fluctuations, and path length fluctuations caused by scattering in the various materials.

Since the muon loses a substantial amount of energy ($\approx 10\,\mathrm{MeV}$) both in TOF1 and TOF2, and they are thick, it would seem to be necessary to understand exactly how and where the timing signal originates. (This needs a bit of thought.)