SciFi Pattern Recognition Error Propagation

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1 Governing equations

The (x, y) projection of a helical particle track is a circle. A circle may be parameterised by three variables. One choice is to use the circle centre coordinates and the radius (x_c, y_c, r).

The path length swept out by a helical track in the (x, y) projection, s, is given by:

$$s = r\phi \tag{1}$$

The turning angle, ϕ, is given by:

$$\phi = \tan^{-1}\left(\frac{y - y_c}{x - x_c}\right) \tag{2}$$

where x, y is a (space-) point on the track.

In order estimate the circle parameters for a set of spacepoints, a linear least squares fit is used with the following alternative parameterisation of the circle (neccessary to linearise the problem):

$$\alpha\left(x^2 + y^2\right) + \beta x + \gamma y + \kappa = 0 \tag{3}$$

The following relations then hold:

$$\alpha = \frac{1}{r^2 - (x_c^2 + y_c^2)} \tag{4}$$

$$\beta = -2x_c\alpha \tag{5}$$

$$\gamma = -2y_c\alpha \tag{6}$$

$$\kappa = -1 \tag{7}$$

These are readily inverted to give:

$$x_c = \frac{-\beta}{2\alpha} \tag{8}$$

$$y_c = \frac{-\gamma}{2\alpha} \tag{9}$$

$$r = \sqrt{\frac{\beta^2 + \gamma^2}{4\alpha^2} - \frac{\kappa}{\alpha}} \tag{10}$$

2 Error propagation to s

The error on s, σ_s, is given by:

$$\sigma_s^2 = \Delta^T V \Delta \tag{11}$$

where V is the covariance matrix, and Δ is a column vector of derivatives of s with respect to the parameters.

2.1 x_c, y_c, r parameterisation

$$V = \begin{pmatrix} \sigma_x^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_y^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_r^2 & \sigma_{rx_c} & \sigma_{ry_c} \\ 0 & 0 & \sigma_{rx_c} & \sigma_{x_c}^2 & \sigma_{x_c y_c} \\ 0 & 0 & \sigma_{ry_c} & \sigma_{x_c y_c} & \sigma_{y_c}^2 \end{pmatrix} \tag{12}$$

$$\Delta = \begin{pmatrix} \partial_{sx} \\ \partial_{sy} \\ \partial_{sr} \\ \partial_{sx_c} \\ \partial_{sy_c} \end{pmatrix} \tag{13}$$

$$\partial_{sx} = \frac{\partial s}{\partial x} = \frac{r\,(y_c - y)}{x_c^2 - 2x_c x + y_c^2 - 2y_c y + x^2 + y^2} \tag{14}$$

$$\partial_{sy} = \frac{\partial s}{\partial y} = \frac{-r\,(x_c - x)}{x_c^2 - 2x_c x + y_c^2 - 2y_c y + x^2 + y^2} \tag{15}$$

$$\partial_{sr} = \frac{\partial s}{\partial r} = \tan^{-1}\left(\frac{y - y_c}{x - x_c}\right) \tag{16}$$

$$\partial_{sx_c} = \frac{\partial s}{\partial x_c} = \frac{r\,(y_c - y)}{x_c^2 - 2x_c x + y_c^2 - 2y_c y + x^2 + y^2} \tag{17}$$

$$\partial_{sy_c} = \frac{\partial s}{\partial y_c} = \frac{r\,(x_c - x)}{x_c^2 - 2x_c x + y_c^2 - 2y_c y + x^2 + y^2} \tag{18}$$

2.2 α, β, γ parameterisation

$$V = \begin{pmatrix} \sigma_x^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_y^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_\alpha^2 & \sigma_{\alpha\beta} & \sigma_{\alpha\gamma} \\ 0 & 0 & \sigma_{\alpha\beta} & \sigma_\beta^2 & \sigma_{\beta\gamma} \\ 0 & 0 & \sigma_{\alpha\gamma} & \sigma_{\beta\gamma} & \sigma_\gamma^2 \end{pmatrix} \tag{19}$$

$$\Delta = \begin{pmatrix} \partial_{sx} \\ \partial_{sy} \\ \partial_{s\alpha} \\ \partial_{s\beta} \\ \partial_{s\gamma} \end{pmatrix} \tag{20}$$

$$\partial_{sx} = -\frac{\left(y + \frac{\gamma}{2\alpha}\right)\sqrt{\frac{1}{\alpha} + \frac{\beta^2+\gamma^2}{4\alpha^2}}}{\left(x + \frac{\beta}{2\alpha}\right)^2 \left(1 + \frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}\right)} \tag{21}$$

$$\partial_{sy} = \frac{\sqrt{\frac{1}{\alpha} + \frac{\beta^2+\gamma^2}{4\alpha^2}}}{\left(x + \frac{\beta}{2\alpha}\right)\left(1 + \frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}\right)} \tag{22}$$

$$\partial_{s\alpha} = \frac{\sqrt{\frac{1}{\alpha} + \frac{\beta^2+\gamma^2}{4\alpha^2}}\left(-\frac{\gamma}{2\alpha^2\left(\frac{\beta}{2\alpha}+x\right)} + \frac{\beta\left(\frac{\gamma}{2\alpha}+y\right)}{2\alpha^2\left(\frac{\beta}{2\alpha}+x\right)^2}\right)}{1 + \frac{\left(\frac{\gamma}{2\alpha}+y\right)^2}{\left(\frac{\beta}{2\alpha}+x\right)^2}} + \frac{\left(-\frac{1}{\alpha^2} - \frac{\beta^2+\gamma^2}{2\alpha^3}\right)\mathsf{tan}^{-1}\left(\frac{\frac{\gamma}{2\alpha}+y}{\frac{\beta}{2\alpha}+x}\right)}{2\sqrt{\frac{1}{\alpha} + \frac{\beta^2+\gamma^2}{4\alpha^2}}} \tag{23}$$

$$\partial_{s\beta} = -\frac{\left(y + \frac{\gamma}{2\alpha}\right)\sqrt{\frac{1}{\alpha} + \frac{\beta^2+\gamma^2}{4\alpha^2}}}{2\alpha\left(x + \frac{\beta}{2\alpha}\right)^2\left(1 + \frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}\right)} + \frac{\beta\mathsf{tan}^{-1}\left(\frac{y+\frac{\gamma}{2\alpha}}{x+\frac{\beta}{2\alpha}}\right)}{4\alpha^2\sqrt{\frac{1}{\alpha} + \frac{\beta^2+\gamma^2}{4\alpha^2}}} \tag{24}$$

$$\partial_{s\gamma} = \frac{\sqrt{\frac{1}{\alpha} + \frac{\beta^2+\gamma^2}{4\alpha^2}}}{2\alpha\left(x + \frac{\beta}{2\alpha}\right)\left(1 + \frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}\right)} + \frac{\gamma\mathsf{tan}^{-1}\left(\frac{y+\frac{\gamma}{2\alpha}}{x+\frac{\beta}{2\alpha}}\right)}{4\alpha^2\sqrt{\frac{1}{\alpha} + \frac{\beta^2+\gamma^2}{4\alpha^2}}} \tag{25}$$

3 Appendix: Mathematica Output

$D[\mathsf{Sqrt}[\alpha\hat{}(-1)+(\beta\hat{}2+\gamma\hat{}2)/(4\alpha\hat{}2)]\mathsf{ArcTan}[(\gamma/(2\alpha)+y)/(\beta/(2\alpha)+x)],x]$

$$-\frac{\left(y+\frac{\gamma}{2\alpha}\right)\sqrt{\frac{1}{\alpha}+\frac{\beta^2+\gamma^2}{4\alpha^2}}}{\left(x+\frac{\beta}{2\alpha}\right)^2\left(1+\frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}\right)}$$

$D[\mathsf{Sqrt}[\alpha\hat{}(-1)+(\beta\hat{}2+\gamma\hat{}2)/(4\alpha\hat{}2)]\mathsf{ArcTan}[(\gamma/(2\alpha)+y)/(\beta/(2\alpha)+x)],y]$

$$\frac{\sqrt{\frac{1}{\alpha}+\frac{\beta^2+\gamma^2}{4\alpha^2}}}{\left(x+\frac{\beta}{2\alpha}\right)\left(1+\frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}\right)}$$

$D[\mathsf{Sqrt}[\alpha\hat{}(-1)+(\beta\hat{}2+\gamma\hat{}2)/(4\alpha\hat{}2)]\mathsf{ArcTan}[(\gamma/(2\alpha)+y)/(\beta/(2\alpha)+x)],\alpha]$

$$\frac{\left(-\frac{\gamma}{2\alpha^2\left(x+\frac{\beta}{2\alpha}\right)}+\frac{\beta\left(y+\frac{\gamma}{2\alpha}\right)}{2\alpha^2\left(x+\frac{\beta}{2\alpha}\right)^2}\right)\sqrt{\frac{1}{\alpha}+\frac{\beta^2+\gamma^2}{4\alpha^2}}}{1+\frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}}+\frac{\left(-\frac{1}{\alpha^2}-\frac{\beta^2+\gamma^2}{2\alpha^3}\right)\mathsf{ArcTan}\left[\frac{y+\frac{\gamma}{2\alpha}}{x+\frac{\beta}{2\alpha}}\right]}{2\sqrt{\frac{1}{\alpha}+\frac{\beta^2+\gamma^2}{4\alpha^2}}}$$

$D[\mathsf{Sqrt}[\alpha\hat{}(-1)+(\beta\hat{}2+\gamma\hat{}2)/(4\alpha\hat{}2)]\mathsf{ArcTan}[(\gamma/(2\alpha)+y)/(\beta/(2\alpha)+x)],\beta]$

$$-\frac{\left(y+\frac{\gamma}{2\alpha}\right)\sqrt{\frac{1}{\alpha}+\frac{\beta^2+\gamma^2}{4\alpha^2}}}{2\alpha\left(x+\frac{\beta}{2\alpha}\right)^2\left(1+\frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}\right)}+\frac{\beta\mathsf{ArcTan}\left[\frac{y+\frac{\gamma}{2\alpha}}{x+\frac{\beta}{2\alpha}}\right]}{4\alpha^2\sqrt{\frac{1}{\alpha}+\frac{\beta^2+\gamma^2}{4\alpha^2}}}$$

$D[\mathsf{Sqrt}[\alpha\hat{}(-1)+(\beta\hat{}2+\gamma\hat{}2)/(4\alpha\hat{}2)]\mathsf{ArcTan}[(\gamma/(2\alpha)+y)/(\beta/(2\alpha)+x)],\gamma]$

$$\frac{\sqrt{\frac{1}{\alpha}+\frac{\beta^2+\gamma^2}{4\alpha^2}}}{2\alpha\left(x+\frac{\beta}{2\alpha}\right)\left(1+\frac{\left(y+\frac{\gamma}{2\alpha}\right)^2}{\left(x+\frac{\beta}{2\alpha}\right)^2}\right)}+\frac{\gamma\mathsf{ArcTan}\left[\frac{y+\frac{\gamma}{2\alpha}}{x+\frac{\beta}{2\alpha}}\right]}{4\alpha^2\sqrt{\frac{1}{\alpha}+\frac{\beta^2+\gamma^2}{4\alpha^2}}}$$