Status of the MICE Optics and Analysis

The MICE collaboration

1 Accelerator Simulation and Optics

1.1 The MICE Muon Beamline

1.1.1 Statistical error on 4D emittance

- Define a 4D emittance in terms of of measured variables x, y, p_x, p_y
- MICE note 451 gives error on measurement of these variables:
  - $\sigma_x = 0.3096\,\mathrm{mm}$
  - $\sigma_y = 0.3143\,\mathrm{mm}$
  - $\sigma_{px} = 1.065\,\mathrm{MeV}$
  - $\sigma_{py} = 1.036\,\mathrm{MeV}$
- Also note that $\sigma_{pz} = 4.117\,\mathrm{MeV}$ and p_z is biased by 2.033 MeV (unclear whether this is + or - from the text)
- Define covariance as $\mathrm{cov}(a, b) = \langle ab\rangle - \langle a\rangle\langle b\rangle$, where $\langle a\rangle$ is the mean value of a, and so on.

The 4D covariance matrix is

$$M = \begin{pmatrix} \mathrm{cov}(x,x) & \mathrm{cov}(x,y) & \mathrm{cov}(x,p_x) & \mathrm{cov}(x,p_y) \\ \mathrm{cov}(y,x) & \mathrm{cov}(y,y) & \mathrm{cov}(y,p_x) & \mathrm{cov}(y,p_y) \\ \mathrm{cov}(p_x,x) & \mathrm{cov}(p_x,y) & \mathrm{cov}(p_x,p_x) & \mathrm{cov}(p_x,p_y) \\ \mathrm{cov}(p_y,x) & \mathrm{cov}(p_y,y) & \mathrm{cov}(p_y,p_x) & \mathrm{cov}(p_y,p_y) \end{pmatrix}, \tag{1}$$

which can be written in simple terms (using $\mathrm{cov}(a, b) = \mathrm{cov}(b, a)$) as,

$$M = \begin{pmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{pmatrix}. \tag{2}$$

Normalised transverse emittance is $\varepsilon_N = \frac{1}{m}\mathrm{det}|M|^{\frac{1}{4}}$, where m is the muon mass.

Assuming the error on m is negligible, the error on the emittance is,

$$\sigma_\varepsilon^2 = (\frac{d\varepsilon}{dM})^2\sigma_m^2, \tag{3}$$

$$\sigma_\varepsilon^2 = \frac{1}{4m}\mathrm{det}|M|^{-\frac{3}{4}}\sigma_M^2, \tag{4}$$

where σ_M is the error on the determinant of the covariance matrix. Working through the algebra for the determinant of a 4D matrix,

$$\mathrm{det}|M| = A\begin{vmatrix} E & F & G \\ F & H & I \\ G & I & J \end{vmatrix} - B\begin{vmatrix} B & F & G \\ C & H & I \\ D & I & J \end{vmatrix} + C\begin{vmatrix} B & E & G \\ C & F & I \\ D & G & J \end{vmatrix} - D\begin{vmatrix} B & E & F \\ C & F & H \\ D & G & I \end{vmatrix} \tag{5}$$

$$\begin{aligned} \mathrm{det}|M| \quad = \quad & +A[E(HJ - I^2) - F(FJ - IG) + G(FI - HG)] \\ & -B[B(HJ - I^2) - F(CJ - ID) + G(CI - HD)] \\ & +C[B(FJ - IG) - E(CJ - ID) + G(CG - FD)] \\ & -D[B(FI - HG) - E(CI - HD) + F(CG - FD)] \end{aligned}, \tag{6}$$

multiplying out and collecting similar terms,

$$\begin{array}{rc} \det|M| = & AEHJ - AEI^2 - AF^2J + 2AFGI - AG^2H \\ & -B^2HJ + B^2I^2 + 2BCFJ - 2BDFI - 2BCGI \\ & +2BDGH - C^2EJ + 2CDEI + C^2G^2 - 2CDFG \\ & -D^2EH + D^2F^2 \end{array} . \tag{7}$$

The error on $\det|M|$, σ_M is given by (assuming the matrix elements aren't correlated between themselves!)

$$\begin{array}{rc} \sigma_M^2 = & \left(\frac{d\det|M|}{dA}\right)^2 \sigma_A^2 + \left(\frac{d\det|M|}{dB}\right)^2 \sigma_B^2 + \left(\frac{d\det|M|}{dC}\right)^2 \sigma_C^2 \\ & + \left(\frac{d\det|M|}{dD}\right)^2 \sigma_D^2 + \left(\frac{d\det|M|}{dE}\right)^2 \sigma_E^2 + \left(\frac{d\det|M|}{dF}\right)^2 \sigma_F^2 \\ & + \left(\frac{d\det|M|}{dG}\right)^2 \sigma_G^2 + \left(\frac{d\det|M|}{dH}\right)^2 \sigma_H^2 + \left(\frac{d\det|M|}{dI}\right)^2 \sigma_I^2 \\ & + \left(\frac{d\det|M|}{dJ}\right)^2 \sigma_J^2 \end{array} , \tag{8}$$

where

$$\frac{d\det|M|}{dA} = EHJ - EI^2 - F^2J + 2FGI - G^2H \tag{9}$$

$$\frac{d\det|M|}{dB} = -HJ + 2BI^2 + 2CFJ - 2DFI - 2CGI + 2DGH \tag{10}$$

$$\frac{d\det|M|}{dC} = -2CEJ + 2DEI + 2CG^2 - 2DFG + 2BFJ - 2BGI \tag{11}$$

$$\frac{d\det|M|}{dD} = -2BFI + 2BGH + 2CEI - 2CFG - 2DEH + 2DF^2 \tag{12}$$

$$\frac{d\det|M|}{dE} = AHJ - AI^2 - C^2J + 2CDI - D^2H \tag{13}$$

$$\frac{d\det|M|}{dF} = -2AFJ + 2AGI + 2BCJ - 2BDI - 2CDG + 2D^2F \tag{14}$$

$$\frac{d\det|M|}{dG} = 2AFI - 2AGH - 2BCI + 2BDH + 2C^2G - 2CDF \tag{15}$$

$$\frac{d\det|M|}{dH} = AEJ - AG^2 - B^2J + 2BGD - D^2E \tag{16}$$

$$\frac{d\det|M|}{dI} = -2AEI + 2AFG + 2B^2I - 2BDF - 2BGC + 2CDE \tag{17}$$

$$\frac{d\det|M|}{dJ} = AEH - AF^2 - B^2H + 2BCF - C^2E \tag{18}$$

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Finally, the errors on the matrix elements $A...J$ must be determined (NB: these do have substantial correlations). Returning to the nomenclature that $\mathrm{cov}(a,b) == \langle ab \rangle - \langle a \rangle \langle b \rangle$ we need to determine the error on the means, $\sigma_{<a>}$ and $\sigma_{<b>}$, and the error on the mean product, $\sigma_{<ab>}$. The error on the means is simply,

$$\begin{array}{rl} \sigma_{<a>} = & \frac{\sigma_a}{\sqrt{N}} \\ \sigma_{<b>} = & \frac{\sigma_b}{\sqrt{N}} \end{array} , \tag{19}$$

where N is the number of particles measured, giving an overall error on the product of the means, $\sigma_{<a><b>}$,

$$\sigma^2_{<a><b>} = \langle b \rangle^2 \sigma^2_{<a>} + \langle a \rangle^2 \sigma^2_{<b>} + 2\langle a \rangle \langle b \rangle \rho \sigma_{<a>} \sigma_{<b>} \tag{20}$$

The error on the mean product is

$$\begin{aligned} \sigma^2_{<ab>} = & \ \frac{1}{N} \sum \left[\left(\frac{d}{da} ab \right)^2 \sigma_a^2 + \left(\frac{d}{db} ab \right)^2 \sigma_b^2 + 2ab\rho\sigma_a\sigma_b \right]. \\ = & \ \frac{1}{N} \sum \left[b^2 \sigma_a^2 + a^2 \sigma_b^2 + 2ab\rho\sigma_a\sigma_b \right] \end{aligned} \tag{21}$$

Then the error on $\mathrm{cov}(a, b)$ is,

$$\sigma_i^2 = \sigma^2_{<ab>} + \sigma^2_{<a><b>}. \tag{22}$$

Working back from the error on the measured parameters, $\sigma_x ... \sigma_{py}$, the emittance and error on the 13923 particles (with a p-value > 0.05 and time-of-flight $> 27\,$ns being the only acceptance criteria applied) from Run 7469 is $\varepsilon_N = 6.205 \pm 0.045$.

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